[1]

NAVIGATION OF SPACETIME SHIPS IN UNIFIED GRAVITATIONAL AND ELECTROMAGNETIC WAVES

JACQUES L. RUBIN
INSTITUT DU NON-LINÉAIRE DE NICE,
U.M.R. 6618 C.N.R.S. - UNIVERSITÉ DE NICE - SOPHIA-ANTIPOLIS,
1361 ROUTE DES LUCIOLES, 06560 VALBONNE, FRANCE
E-MAIL: JACQUES.RUBIN@INLN.CNRS.FR

Abstract. On the basis of a *"local"* principle of equivalence of general relativity, we consider a navigation in a kind of "4D-ocean" involving measurements of conformally invariant physical properties only. Then, applying the Pfaff theory for PDE to a particular conformally equivariant system of differential equations, we show the dependency of any kind of function describing "spacetime waves", with respect to 20 parametrizing functions. These latter, appearing in a linear differential Spencer sequence and determining gauge fields of deformations relatively to "ship-metrics" or to "flat spacetime ocean metrics", may be ascribed to unified electromagnetic and gravitational waves. The present model is based neither on a classical gauge theory of gravitation or a gravitation theory with torsion, nor on any Kaluza-Klein or Weyl type unifications, but rather on a post-Newtonian approach of gravitation in a four dimensional conformal Cosserat spacetime.

Key Words: Conformal invariance, Cosserat space, differential sequences, electromagnetism, gauge theory, gravitation, model of unification, Pfaff systems, spacetime, Spencer theory of partial differential equations.

PACS-2001 Subject Classification: Primary 04.20.-q Classical general relativity, 12.10.-g Unified field theories and models; Secondary 02.30.Jr Partial differential equations.

Running Title: "Spacetime Ships in a 4D-Ocean"

[1]This is the second version (v.3) of the gr-qc/0205012 paper. Relative to the second version, some changes in the mathematical results have been done without consequences on the physical model (see also hep-th/0404186). The conformally flatness of the substratum spacetime, i.e. the vanishing of the Weyl tensor associated to the metric $\underline{\omega}$, which is an assumption used throughout in the mathematical developments from chapter 2, has been well precised in the first chapter. Clearer explanations at the very end of chapter 3 about accelerating frames are given. New references are indicated and some of them corrected.

Date: Preprint INLN 2002/13, February 7, 2008, v.3.

1. Introduction: spacetime as a 4D-ocean

1.1. **Smooth and striated spaces.** Contrarily to what is usually known about the ether theory, A. Einstein wasn't opposed to this very concept, but rather to the concept of a favored frame and to a concept of ether considered as a rigid object necessary to the propagation of light. After rejecting the ether, he finally accepted it on the basis of the following assumptions given in *"Aether und Relativitaetstheorie"* (Julius Springer Ed., Berlin, 1920, 15 pages. From a lecture held on 5th may 1920 at Leiden University):

> p. 8: *"Der nächstliegende Standpunkt, den man dieser Sachlage gegenüber einnehmen konnte, schien der folgende zu sein. Der Äther existiert überhaupt nicht."* (The most obvious viewpoint which could be taken of this matter appeared to be the following. The ether does not exist at all.)
>
> p. 9: *"Indessen lehrt ein genaueres Nachdenken, daß diese Leugnung des Äthers nicht notwendig durch das spezielle Relativitätsprinzip gefordert wird. Man kann die Existenz eines Äthers annehmen; nur muß man darauf verzichten, ihm einen bestimmten Bewegungszustand zuzuschreiben, d. h. man muß ihm durch Abstraktion das letzte mechanische Merkmal nehmen, welches ihm Lorentz noch gelassen hatte."* (However, closer reflection shows that this denial of the ether is not demanded by the special principle of relativity. We can assume the existence of an ether, but we must abstain from ascribing a definitive state of motion to it, i.e. we must by abstraction divest it of the last mechanical characteristic which Lorentz had left it.)
>
> p. 10: *"Verallgemeinernd müssen wir sagen. Es lassen sich ausgedehnte physikalische Gegenstände denken, auf welche der Bewegungsbegriff keine Anwendung finden kann.... Das spezielle Relativitätsprinzip verbietet uns, den Äther als aus zeitlich verfolgbaren Teilchen bestehend anzunehmen, aber die Ätherhypothese an sich widerstreitet der speziellen Relativitätstheorie nicht. Nur muß man sich davor hüten, dem Äther einen Bewegungszustand zuzusprechen."* (Generalizing, we must say that we can conceive of extended physical objects to which the concept of motion cannot be applied.... The special principle of relativity forbids us to regard the ether as composed of particles, the movements of which can be followed out through time, but the ether hypothesis as such is not incompatible with the special theory of relativity. Only we must take care not to ascribe a state of motion of the ether.)

Moreover, denying ether amounts to consider spacetime as deprived of any physical properties, which is obviously not the case. This would suggest a concept of ether as a medium referring to physical properties and not to geometric or mechanical considerations (see p. 12 in the reference above). Somehow adopting this point of view, we will consider ether as a "4D-*ocean*" and later on, as a further specification, as a *"spacetime ocean"*. We recall that this analogy has been made few years ago by W. G. Unruh [1], and detailed discussions of the concepts involved in ether theory can be found in [2, 3].

At first, in order to introduce as clearly as possible our model of unification of electromagnetic and gravitational forces, we will rely on metaphors in two and three dimensions.

Suppose a child asks you: *"At which distance the blue sky is ?"*. Or likewise on a "sailboat" looking afar at a ship on a quiet blue ocean: *"How far is that ship ?"*. May be in a future orbital station, another child will wonder: *" Why can't we touch the stars with our hands ?"*.

All of these apparently naive questions send us back to one only but major difficulty: the ocean, the sky, the outer space may be topological, still surely non-metrical spaces (!), taking "metrical" in its usual acceptation. In the case of a spacetime ocean, assumed to be a non-metrical space, the latter remark forbids to conceive any kind of state of spacetime motions, since a notion of distance would be required to evaluate, for instance, the spacetime velocity. We know how difficult it is to evaluate distances on sea, or altitudes of aircraft just by looking at them (or evaluate times without watches in spacetime). These kinds of spaces are not *"striated"* ones as our highways with permanent blank dashed lines on ground allowing for evaluations of distances: the former are not "naturally" endowed with fields of metrics. The coordinate maps, from these spaces to $\mathbb{R}$-vector spaces, are only defined on those subspaces of points at which serial physical measurement processes are performed (i.e. attributions of finite sets of numbers to some points or some composed parts). Distances could be undefined on those subspaces or, at best, only defined on them. In fact, we might have only a metric attached to each point of a line, a trajectory of a ship, the wake of an airplane ... and this metric would be built out of a local moving frame. The latter could be the wings arrow of the airplane, the mast and the boom of a sailboat, but also their wakes, furnishing a velocity vector or a flux vector. In fact, spacetime has a definite number of geometric dimensions but the latter cannot be attributed *"a priori"* to space or time dimensions: this ascription of an orientation can be performed only *locally* in the moving frames.

Angle measurements only can be achieved with a calliper on a kind of "sea horizon circle", or with a sextant on the well-known "celestial sphere": Now these are just the elements of a conformal geometry. Of course these operations are strongly related to our eyesight and the light, and it has been demonstrated in two classical 1910 studies of H. Bateman [4] and E. Cunningham [5] that the Maxwell equations of electromagnetic fields in vacuum are conformally equivariant (see also [6]). In sea or aerial navigation, from known beacons or hertzian markers out of which angles can be measured, we can deduce the geometric positions on charts. But, if just after, we lose the signals for a while (because of fog or solar storms hiding the coast or perturbating electromagnetic signals from markers), then we need to redirect our way in order to recover them. But how this can be done without the use of magnetic compasses or gyroscopes to orient angles, and then to know the north, the south, the bottom and the top ?

Clearly we need magnetic and gravitational forces to orient the moving frames or the angles, and this means that, to be known, local geometry requires that forces be there (!), to provide at least directions. These 2D and 3D spaces must be endowed with a (magnetic or gravitational) field of orientation. On that point, extending this approach to a 4D-ocean, we should have such a force to discriminate between past and future (which doesn't mean to have an universal duration), allowing for instance to orient the light cones. In some way, we would have to consider a "time" force vector "dressing" the "spacetime",

and assume a field of time orientation added to the spacetime structure. We will come back on that point in the conclusion since it is really a major historic difficult question.

It follows local geometries are deduced from forces and not the converse as is usually done in general relativity, which, in this respect, appears to be in conflict with the principles of navigations. We take the navigation side and the "Prima" of the forces on the geometry. Nevertheless, against appearances, the *"local"* principle of equivalence of general relativity will be kept throughout, but used another way.

Here we present two kinds of "spaces" to which will be given synonyms depending on the context (metaphorical, mathematical or physical):

- a *"smooth space"* that will be sometimes referred to as the *"unfolded spacetime"*, the "4D-*ocean"* or the *"spacetime ocean"* and denoted by $\mathcal{M}$, and
- a *"striated space"* also dubbed the *"underlying or substratum spacetime"* or the *"*4D-*ocean ground floor* $\mathcal{S}$*"*.

Our *"ships"*, *"sailboats"*, *"aircraft"* would be the tangent spaces $T_{p_0}\mathcal{M}$ or moving frames, also viewed as the space of rulers, callipers and watches. As a matter of fact, this model will involve an unfolding or a deployment of the *"smooth space"* $\mathcal{M}$ from the *"striated space"* $\mathcal{S}$ (we use the G. Deleuze and F. Guattari terminology [7, §12 and §14]). It may also be viewed as a kind of generalisation in the PDE framework, of the universal deployment concept for ODE introduced in particular by the mathematician René Thom in his well-known catastrophes theory. The relevant modern mathematical terminology would be: cobordism theory.

1.2. **To tie a spacetime ship with its environnement: the principle of equivalence.** Let us assume the unfolded spacetime $\mathcal{M}$ to be of class C^∞, of dimension $n \geq 4$, connected and paracompact. Let p_0 be a particular point in $\mathcal{M}$, $U(p_0)$ an open neighborhood of p_0 in $\mathcal{M}$, and $T_{p_0}\mathcal{M}$ its tangent space. The *"local"* principle of equivalence we use (compatible with the usual ones. See a review of those principles in [8]), states it exists a local diffeomorphism φ_{p_0} attached to p_0 putting in a one-to-one correspondence the points $p \in U(p_0)$ with some vectors $\xi \in T_{p_0}\mathcal{M}$ in an open neighborhood of the origin of $T_{p_0}\mathcal{M}$:

$$\varphi_{p_0} : p \in U(p_0) \subset \mathcal{M} \longrightarrow \xi \in T_{p_0}\mathcal{M}\,, \qquad \varphi_{p_0}(p_0) = 0\,.$$

It is important to remark the correspondence between "position points" and "position vectors". We will show in the next chapter it involves the Einstein's principle of equivalence between relative uniformly accelerated frames. To each position vector ξ of $T_{p_0}\mathcal{M}$, we can associate a frame made out of a "little" local web of straight lines and thus we construct a *local* field of metrics $\bar{g}_{p_0}(\xi)$ depending both on p_0 and the point ξ. This field of metrics is defined on the tangent spaces of the tangent space: $T_\xi(T_{p_0}\mathcal{M})$. We will further assume that these webs are (partially) oriented with respect to the *local* orientation provided by a time arrow (like the needle of a magnetic compass in 2D). Hence a particular direction is selected among the four, and reflected in the signature of the metric field $\bar{g}_{p_0}$ assumed to be of the Minkowski type $(+\,-\,-\,-)$ as "usual".

To proceed further, let us evoke a metaphor borrowed to the Quattrocento painters: considering landscapes ($\mathcal{S}$), they wished to draw them on canvas ($T_{p_0}\mathcal{M}$), seeing them through perspective grids or webs ($\mathcal{M}$). In a 4D situation the "grids" would deform

themselves and without an absolute grid of reference we may think instead of a dynamical deformation of the landscapes.

More precisely, we consider other "little" webs (at $p \in U(p_0) \subset \mathcal{M}$) on the "surface" of the 4D-ocean $\mathcal{M}$. If we try to superpose the latter with those at $\xi \in T_{p_0}\mathcal{M}$ so as to simultaneously see their "images", we make a projective conformal correspondence, as painters did with their geometric perspectives. Moreover if a *local* metric $\tilde{g}$ depending only on $p \in \mathcal{M}$ is attached to this latter frame web, this correspondence means that the metric $\bar{g}_{p_0}$ is pulled back by the application $\varphi^*_{p_0}$ to $\tilde{g}$. Then we have the relation: $\varphi^*_{p_0}(\bar{g}_{p_0}) = \tilde{g}$, and the metrics $\tilde{\omega}$ and $\bar{\omega}$ on $\mathbb{R}^n$ of type $(+\,-\,-\,-)$ associated respectively to $\tilde{g}$ and $\bar{g}_{p_0}$ should be thought of as being somehow proportional in view of the previous metaphorical perspective. In other words they are conformally equivariant with respect to *local* diffeomorphisms $\hat{f}_0$ of $\mathbb{R}^n$ depending on p_0 and associated to φ_{p_0}, i.e. conformally equivariant with respect to the conformal Lie pseudogroup associated with a metric field μ defined on $\mathbb{R}^n$, of type $(+\,-\,-\,-)$. Thus we have in this case:

$$\hat{f}^*_0(\bar{\omega}) = \tilde{\omega}\,, \qquad \tilde{\omega} \simeq_{p_0} e^{2\lambda_0}\,\bar{\omega} \equiv \nu\,, \qquad \hat{f}_0(0) = 0\,, \qquad \lambda_0(0) = 0\,, \tag{1}$$

and

$$\hat{f}^*_0(\mu) = e^{2\alpha_0}\,\mu\,, \qquad \alpha_0(0) = 0\,, \tag{2}$$

where λ_0 *and* α_0 *are local functions associated to each* $\hat{f}_0$ and also depending on p_0. Also we indicate with the sign "$\simeq_{p_0}$" the *local* equality at or for a given point p_0. In this expression we point out that λ_0 *and* α_0 *are functions* in full generality. *It is an essential feature of the model under consideration.* In case of constants, then 15 parameters would define $\hat{f}_{p_0}$ and it is known that nothing could be ascribed to electromagnetism but only to a uniform gravitational field.

The metric $\tilde{g}$ emerges also from an other outlook but in the same framework. Indeed we can consider the *"susbstratum spacetime $\mathcal{S}$"* as a striated space, that is, a metrical space (a metaphorical description might consist in referring to an ocean ground floor made out of *striated sand*). Thus, we assume it can be endowed with a *global* metric we can denote by $\underline{g}$. Also we make the assumption that $\mathcal{S}$ has a constant Riemaniann curvature tensor and then is "conformally flat", i.e. the Weyl tensor is vanishing. This will ensure the integrability of the conformal Lie pseudogroup from the H. Weyl theorem [9]. In other words, the webs "drawn" on $\mathcal{S}$ are only transformed into each others, at the same ground level, with no changes in the "perspective": no dilatation transformations occur. Hence this "substratum spacetime $\mathcal{S}$" is equivariant with respect to *global* diffeomorphisms ψ of the Poincaré Lie pseudogroup associated to a *global* metric field denoted $\underline{g}$ and satisfying, for every *global* diffeomorphism ψ of $\mathcal{S}$, $\psi^*(\underline{g}) = \underline{g}$. If $\underline{\omega}$ represents $\underline{g}$ on $\mathbb{R}^n$, and if the global diffeomorphims ψ are associated to global applications f of the Poincaré Lie pseudogroup of $\mathbb{R}^n$, endowed again with the metric field μ, then we have

$$f^*(\underline{\omega}) = \underline{\omega}\,. \tag{3}$$

Continuing with the metaphorical description, we try to superpose again the web at $p \in \mathcal{M}$, and a web "drawn" on $\mathcal{S}$. Each one is just a mathematically *smooth* but, a priori at that step, a completly general deformation of the other. As a consequence, it exists

local diffeomorphisms ϕ_p such that $\phi_p^*(\underline{g}) = \tilde{g}$. Then from the latter and expression (1), we have on $\mathbb{R}^n$, the relation

$$h_p^*(\underline{\omega}) = \bar{\omega}\,, \qquad h_p(0) = 0\,, \tag{4}$$

where h_p is a *local* diffeomorphism of $\mathbb{R}^n$. *Then we make the simplification assumption: the metric field μ is identified with $\underline{\omega}$, so that one obtains from* (2) *the relation:*

$$\hat{f}_0^*(\underline{\omega}) = e^{2\alpha_0}\underline{\omega}\,. \tag{5}$$

But also from relations (1) and (4), we can deduce the following commutative diagram:

$$\begin{tikzcd}
\underline{\omega} \arrow[r, "f^*"] \arrow[d, "h_p^*"'] & \underline{\omega} \arrow[d, "e^{2\lambda_0} h_p^*"] \\
\bar{\omega} \arrow[r, "\hat{f}_0^*"'] & \nu
\end{tikzcd}$$

Therefore, to pass from $\underline{\omega}$ to $\bar{\omega}$ (or from a point $\underline{p} \in \mathcal{S}$ to a "point of deformation" $\xi \in T_{p_0}\mathcal{M}$) is equivalent to pass from f to $\hat{f}_0$.

Unfortunately (!), all the measurements are performed at p_0. That means we only consider angles as well as the metric field $\bar{\omega}$ up to any multiplicative factor, i.e. we must consider the metric on the projective 3-dimensional space attached to p_0, namely $\mathbf{P}\mathbb{R}^{n-1}$. Hence only the diffeomorphisms $\hat{f}_0$ giving any metric field ν different from $\bar{\omega}$ (or equivalently $\underline{\omega}$) will matter for physical applications as we will see later on.

To summarize, to pass from the substratum spacetime $\mathcal{S}$ with the metric field $\underline{\omega}$ to the tangent spacetime $T_{p_0}\mathcal{M}$ with the metric field $\bar{\omega}$ is equivalent to pass from the Poincaré Lie pseudogroup to the conformal one, the two being associated with $\underline{\omega}$. Hence this deformation process is associated to the coset of the two previous pseudogroups. It is a set of functions parametrizing all the deformation diffeomorphisms h. These functions are strongly related to the arbitrary functions family α_0 and λ_0, as well as their derivatives. And we will show that they can be related to the electromagnetic and gravitational gauge potentials in a unified way.

Of course, this very classical approach in deformation theory differs from the classic gauge one in general relativity (see a review for instance in [10]). Indeed the latter are developed from a given gauge Lie group, which can be Poincaré, conformal, or almost any other Lie group. But first, they are not pseudogroups, and in second place they are assossiated to Lie groups invariance of the tangent spaces (not the tangent fiber bundle) at any fixed base point p_0. In fact that means they are isotropy Lie subgroups of the corresponding pseudogroups, or equivalently Lie groups of the fibers of the tangent bundles, namely structural Lie groups. For instance, in fixing the function α_0 to a constant, meaning fixing p_0, the set of applications $\hat{f}_0$ becomes a Lie group and not a Lie pseudogroup. In that case the applications $\hat{f}_0$ would depend only on 15 real variables (if $n = 4$), not on a set of arbitrary functions as we will see for the conformal pseudogroup.

Also it is not a Kaluza-Klein type theory since we just consider a spacetime with dimension $n = 4$, even though the results, we present here, can be extended to higher

dimensions n. The fields of interactions don't come from the addition of extra geometrical dimensions other than those necessary to the geometrical description of our four dimensional physical spacetime.

Sometimes an unification is suggested in considering a model based on a possible torsion of an affine connection, i.e. based on a Riemann-Cartan geometry. In the present model, the torsion is merely associated to motions of ships, aircraft, cars such as pitching, rolling, precession, rotation, etc.... The skew-symmetric part of the connection symbols such as the Riemann-Christoffel ones (no torsion) is related to these latter kinds of motions and not to a sort of Faraday tensor. In fact these motions are only associated to kinds of constraints (such an example is provided by contraints of Lorentz invariance in the Thomas precession mechanism) between a vector η and its tangent space and not to the base space $\mathcal{M}$, i.e. the environment with its "4D-waves".

Also it differs completly from H. Weyl unifications and the J.-M. Souriau approach [11]. In fact there were two different theories proposed by H. Weyl, in 1918 and latter on in 1928-29 [12, 13]. These theories were based on variational equivariance with respect to local *given* dilatation transformations, exhibiting some kinds of Noether invariants ascribed to electric charges or electric currents. In some way, the model we present is also related to dilatation transformations but not to a variational problem and/or to topological invariants. Moreover, our model deals precisely with the determination of space of dilatation operations that are compatible with the conformal transformations preserving the equivariance a given metric field. As should become clear in next chapter 3, not all dilatations are permitted, since they are bounded by constraints reflected in particular functional dependencies. The latter will reflect themselves through the existence of parametrizing functions which will be physically ascribed to fields or gauge potentials of interactions or of spacetime deformations.

Close approaches to ours, are developed on the one hand by M. O. Katanaev & I. V. Volovich [14], and on the other hand by H. Kleinert [15] and J.-F. Pommaret [16]. These authors, however, do not seem to have been concerned with any model of unification, since a particular system of PDE presented in the sequel, namely the "c system", is not considered at the roots of their models. Our model, in the end, comes out pretty much in line with their approach of general relativity in Cosserat media or space. Other general relativity models at lower dimensions was investigated with such approaches as models of gravity in [17, 18, 19] for instance.

Also, our present work can be viewed somehow as a kind of generalizing extension of the T. Fulton *et al.* approach and model [6].

Here below, we summarize the mathematical procedure and assumptions presented in the sequel and based on the previous discussion (we refer to definitions of involution i.e. integrability, symbols of differential equations, acyclicity and formal integrability such as those given in [20, 21, 16] *for instance):*

- *The Riemann scalar curvature ρ_s associated to the metric field $\underline{\omega}$ hereafter denoted by ω, is a constant, $n(n-1)k_0$, as a consequence of the constant Riemann curvature tensor assumption. In that case, let us recall that the Poincaré Lie pseudogroup is involutive (i.e. integrable) since, on the one hand, its symbol of order 2 vanishes*

which makes it n-acyclic, and on the other hand, it is formally integrable if $\rho_{\rm s}$ is a constant. Then, the Weyl tensor associated to $\underline{\omega}$ is vanishing, i.e. we have a conformally flat structure.

- *The system* (5) *of differential equations in $\hat{f}_0$, hereafter denoted by $\hat{f}$, being non-integrable (i.e. it exists non-analytic C^∞ solutions on some open neighborhoods), will be supplied with an other system of equations (the system we call "c system" in the sequel)), obtained from a prolongation procedure which will be stopped as the integrability conditions of the resulting complete system of partial differential equations is met (i.e. all C^∞ solutions are analytic almost everywhere).*
- *The covariant derivations involved in the prolongation procedure will be assumed to be torsion free, i.e. we will make use of the Levi-Civita covariant derivations.*
- *We will extract from the latter system of PDE, a subsystem, which will be called the "c system", defining completely the sub-pseudogroup $\Gamma_{\widehat{\rm H}}$ of those applications $\hat{f}$ which are strictly smooth deformations of applications f. This PDE subsystem will be satisfied by a function α_0, hereafter denoted by α.* This is the core system of our model and to our knowledge it has never been really studied or at least related to any unification model.
- *By considering Taylor series solutions of the "c system", and searching for conditions under which an $\mathbb{R}$-valued Taylor serie $s(x, x_0)$ around a point $x_0 \in \mathbb{R}^n$ does not depend on x_0, we will show how general solutions depend on a particular finite set of parametrizing functions.*
- *We show that this set of parametrizing functions is associated to a differential sequence, and that they can be identified with both electromagnetic and gravitational gauge potentials. Moreover, they obey a set of PDE called the "first group of PDE".* We thus get a gauge theory, similar to the electromagnetic theory, though not based on the de Rham sequence, but rather on a kind of linear Spencer sequence for non-linear PDE *(see the general linear sequences for the linear PDE case in* [20]*).*
- *We deduce the metric field $\nu \equiv \omega + \delta\omega$ of the infinitesimal smooth deformations, depending on the electromagnetic and gravitational gauge potentials.*
- *We give general Lagrangian densities associated to ν, and in particular we show how the occurence of a spin property makes it possible to define an electromagnetic Lagrangian density, sheding some new light on the relation between spin and electromagnetic interactions with charged particules.*
- *We conclude with various remarks about physical interpretations of general relativity.*

To finish this chapter, we indicate that the mathematical tools used for this unification finds its roots, first in the conformal Lie structure that has been extensively studied by H. Weyl [9], K. Yano [22], J. Gasqui [23] and J. Gasqui & H. Goldschmidt [21] for instance, and in second place, in the non-linear cohomology of Lie equations studied by B. Malgrange [24, 25], A. Kumpera & D. Spencer [26] and J.-F. Pommaret [16]. Meanwhile we only partially refer to some of these aspects since it mainly has to do with the general theory of Lie equations, and not exactly with the set of PDE we are concerned with. Indeed these set of PDE is not the set of conformal Lie equations themselves, but rather a kind of "residue" coming from a comparison with "Poincaré Lie equations", i.e. the "c

system". Also, a large amount of mathematical results have been obtained and complete reviews exist, that are devoted to conformal geometries [27]. Hence, most of the results concerning this geometry are summarized in the sequel. We essentially indicate succintly the cornerstones which are absolutely necessary for our explanations and descriptions of the model.

2. The conformal finite Lie equations of the substratum spacetime

First of all, and from the previous sections, we assume that the group of relativity is no longer Poincaré but the conformal Lie group. In particular, this involves that no physical law changes occur shifting from a given frame embeded in a gravitational field to a uniformly accelerated relative isolated one, as we will recall [28]. This is just the elevator metaphor at the origin of the Einstein's principle of equivalence.

The conformal finite Lie equations are deduced from the conformal action on a *local* metric field ω defining a pseudo-Riemannian structure on $\mathbb{R}^n \simeq \mathcal{S}$. We insist on the fact we do local studies, meaning we consider *local* charts from open subsets of the latter manifolds into a common open subset of $\mathbb{R}^n$. Hence by geometric objects or computations on $\mathbb{R}^n$, we mean *local* geometric objects or computations on the manifolds $\mathcal{M}$, $T\mathcal{M}$ and $\mathcal{S}$. Also, it is well-known that the mathematical results displayed below are independent of the dimension when greater or equal to 4. Let us consider $\hat{f} \in Diff^{\infty}_{\mathrm{loc.}}(\mathbb{R}^n)$, the set of local diffeomorphisms of $\mathbb{R}^n$ of class C^∞, and any function $\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Then if $\hat{f} \in \Gamma_{\widehat{\mathrm{G}}}$ ($\Gamma_{\widehat{\mathrm{G}}}$ being the pseudogroup of local conformal bidifferential maps on $\mathbb{R}^n$), $\hat{f}$ is a solution of the PDE system (in fact other PDE must be satisfied to completely define $\Gamma_{\widehat{\mathrm{G}}}$ as one will be seen in the sequel):

$$\hat{f}^*(\omega) = e^{2\alpha}\,\omega\,, \tag{6}$$

with $\det(J(\hat{f})) \neq 0$, where $J(\hat{f})$ is the Jacobian of $\hat{f}$, and $\hat{f}^*$ is the pull-back of $\hat{f}$. Also, only the $\mathrm{e}^{2\alpha}$ positive functions are retained in view of the previous assumption, that only one orientation is chosen on $\mathbb{R}^n$. We therefore consider only the $\hat{f}$'s which preserve that orientation. We recall that α *is a varying function depending on each* $\hat{f}$ *and consequently not fixed*. We denote $\tilde{\omega}$ the metric on $\mathbb{R}^n$ such as by definition: $\tilde{\omega} \simeq \mathrm{e}^{2\alpha}\omega$, and we agree on putting a tilde on each tensor or geometrical "object" relative to, or deduced from this metric $\tilde{\omega}$.

Now, performing a first prolongation of the system (6), we deduce other second order PDE's connecting the Levi-Civita covariant derivations ∇ and $\widetilde{\nabla}$, respectively associated to ω and $\tilde{\omega}$. These new differential equations are (see for instance [23]) $\forall\, X,\, Y \in T\mathbb{R}^n$:

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\alpha(X)Y + d\alpha(Y)X - \omega(X,Y)\,{}_*d\alpha\,, \tag{7}$$

where d is the exterior differential and ${}_*d\alpha$ is the dual vector field of the 1-form $d\alpha$ with respect to the metric ω, i.e. such that $\forall\, X \in T\,\mathbb{R}^n$:

$$\omega(X, {}_*d\alpha) = d\alpha(X)\,. \tag{8}$$

Since the Weyl tensor τ associated to ω is assumed to vanish, the Riemann tensor ρ can be rewritten $\forall\, X, Y, Z, U \in C^\infty(T\mathbb{R}^n)$ as:

$$\omega(U, \rho(X,Y)Z) = \frac{1}{(n-2)}\left\{\omega(X,U)\sigma(Y,Z) - \omega(Y,U)\sigma(X,Z) \right. \\ \left. + \,\omega(Y,Z)\sigma(X,U) - \omega(X,Z)\sigma(Y,U)\right\}, \quad (9)$$

where σ is defined by (see the tensor "L" in [22] up to a constant depending on n)

$$\sigma(X,Y) = \rho_{\mathrm{ic}}(X,Y) - \frac{\rho_{\mathrm{s}}}{2(n-1)}\omega(X,Y), \quad (10)$$

where ρ_{ic} is the Ricci tensor and ρ_{s} is the Riemann scalar curvature. Consequently, the first order system of PDE in $\hat{f}$ "connecting" $\tilde{\rho}$ and ρ, can be rewritten as a first order system of PDE concerning $\tilde{\sigma}$ and σ. Using the torsion free property of the Levi-Civita covariant derivations, one obtains the *third* order system of PDE (since α is depending on the first order derivatives of $\hat{f}$):

$$\hat{f}^*(\sigma)(X,Y) \equiv \tilde{\sigma}(X,Y) = \sigma(X,Y) + (n-2)\Big(d\alpha(X)d\alpha(Y) \\ - \frac{1}{2}\omega(X,Y)d\alpha({}_*d\alpha) - \mu(X,Y)\Big), \quad (11)$$

in which we have defined the symetric tensor $\mu \in C^\infty(S^2\mathbb{R}^n)$ by:

$$\mu(X,Y) = \frac{1}{2}\left[X(\,d\alpha(Y)) + Y(d\alpha(X)) - d\alpha(\nabla_X Y + \nabla_Y X)\right]. \quad (12)$$

To go further, it is important again to notice that the relation (11) is directly related to a third order system of PDE we denote (T), since it is deduced from a supplementary prolongation procedure applied to the second order system (7). Then it follows, from the well-known theorem of H. Weyl on equivalence of conformal structures [9, 22, 21], and because of the Weyl tensor vanishing, that the systems of differential equations (6) and (7) when completed with the latter third order system (T), becomes an involutive system of order three. Let us stress again that α is merely defined by $\hat{f}$ and its first order derivatives, according to the relation (6).

Looking only at those applications $\hat{f}$ which are smooth deformations of applications f, this third order system of PDE must reduce to a particular conformal Lie sub-pseudogroup we denote by $\Gamma_{\widehat{\mathrm{H}}}$. Indeed, if α tends towards the zero function with respect to the C^2-topology, then the previous set of smoothly deformed applications $\hat{f}$ must tend towards the Poincaré Lie pseudogroup. But this condition is not satisfied by all of the application $\hat{f}$ in the conformal Lie pseudogroup $\Gamma_{\widehat{\mathrm{G}}}$, since in full generality, the non-trivial third order system of PDE (T) would be kept at the zero α function limit. In other words, the pseudogroup $\Gamma_{\widehat{\mathrm{H}}}$ could be defined by an involutive second order system of PDE which would tend towards the involutive system defining the Poincaré Lie pseudogroup.

The systems of differential equations (6) and (7) would be well suited to define partially this pseudogroup $\Gamma_{\widehat{\mathrm{H}}}$. The n-acyclicity property of $\Gamma_{\widehat{\mathrm{H}}}$ would be restored and borrowed, at the order two, from the Poincaré one, provided however that an, a priori, arbitrary *input* perturbative function α is given before, instead of being defined from an

application $\hat{f}$ according to the relation (6). But the formal integrability is obtained only if the tensor σ satisfies one of the following equivalent relations (see formula (16.3) with definition (3.12) in [21]):

$$\sigma = k_0 \frac{(n-2)}{2} \omega \quad \Longleftrightarrow \quad \rho_{\text{ic}} = (n-1)\, k_0\, \omega\,, \tag{13}$$

deduced from relation (10), in order to avoid adding up the supplementary first order system of PDE (11) to (6). Then, considering the system (6), the system (11) reduces to a second order system of PDE concerning only the input function α, which is thus constrained, contrarily to what might have been expected, and such that:

$$\mu(X,Y) = \frac{1}{2} \left\{ \left[k_0 \left(1 - \mathrm{e}^{2\alpha}\right) - d\alpha({}_*d\alpha) \right] \omega(X,Y) \right\} + d\alpha(X) d\alpha(Y)\,. \tag{14}$$

Obviously, as can be easily verified, this is an involutive system of PDE as it can be easily verified, since it is a formally integrable system with a vanishing symbol (i.e. elliptic symbol) of order two.

Thus, we have series of PDE deduced from (6) defining all the *smooth* deformations of the applications f contained in the conformal Lie pseudogroup $\Gamma_{\widehat{\mathrm{G}}}$.

In addition, the metric field ω necessarily satisfies relation (13) (which is analogous to the Einstein equations but with a stress-energy tensor proportional to the metric one) if the Lie pseudogroup $\Gamma_{\widehat{\mathrm{H}}}$, containing the Poincaré one, strictly differs from the latter. In that case, such a metric field ω and the substratum spacetime $\mathcal{S}$ will be called *S-admissible* (with S as "Substratum"), and this S-admissibility is assumed to be satisfied in the sequel.

In an orthonormal system of coordinates, the PDE (6), (7) and (14) defining $\Gamma_{\widehat{\mathrm{H}}} \subset \Gamma_{\widehat{\mathrm{G}}}$ can be written, with $\det(J(\hat{f})) \neq 0$ and $i, j, k = 1, \cdots, n$ as:

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{f})\ \hat{f}^r_i \hat{f}^s_j = e^{2\alpha} \omega_{ij}\,, \tag{15a}$$

$$\hat{f}^k_{ij} + \sum_{r,s=1}^{n} \gamma^k_{rs}(\hat{f})\ \hat{f}^r_i \hat{f}^s_j = \sum_{q=1}^{n} \hat{f}^k_q \left(\gamma^q_{ij} + \alpha_i \delta^q_j + \alpha_j \delta^q_i - \omega_{ij} \alpha^q \right)\,, \tag{15b}$$

$$\mu_{ij} = \alpha_{ij} - \sum_{k=1}^{n} \alpha_k \gamma^k_{ij} = \frac{1}{2} \Big\{ k_0 (1 - \mathrm{e}^{2\alpha}) - \sum_{k=1}^{n} \alpha^k \alpha_k \Big\} \omega_{ij} + \alpha_i \alpha_j\,, \tag{15c}$$

where δ^i_j is the Kronecker tensor, and where one denotes as usual $\hat{f}^i_j \equiv \partial \hat{f}^i / \partial x^j \equiv \partial_j \hat{f}^i$, etc ..., $T_k = \sum_{h=1}^{n} T^h\, \omega_{hk}$ and $T^k = \sum_{h=1}^{n} T_h\, \omega^{hk}$ for any tensor T where ω^{ij} is the inverse metric tensor, and γ is the Riemann-Christoffel form associated to the *S-admissible* metric ω. This is the set of our starting equations. It matters to notice that the (T) system is not included in the above set of PDE. Indeed, the latter being already involutive from order 2, this involves from definition of involution that the (T) system is redundant since all the applications $\hat{f} \in \Gamma_{\widehat{\mathrm{H}}}$, solutions of (15), will also be solutions of all the systems of PDE obtained by prolongation.

Under a change of coordinates with a conformal application $\hat{f}$ and assuming a constant metric field ω, the function $\alpha (\equiv \tilde{\alpha}_0)$ and the tensor $\tilde{\alpha}_1 \equiv \{\alpha_1, \cdots, \alpha_n\}$ are transformed

into "primed" functions and tensors such as ($j = 1, \cdots, n$):

$$\alpha'(\hat{f}) = \alpha - \frac{1}{n} \ln |\det J(\hat{f})| , \qquad \sum_{i=1}^{n} \hat{f}^i_j \, \alpha'_i(\hat{f}) = \alpha_j - \sum_{k,l=1}^{n} \frac{1}{n} \, (\hat{f}^{-1})^k_l \circ \hat{f} \;\, \hat{f}^l_{kj} \, ,$$

which essentially displays the affine structure of these "geometrical objects". In particular, the tensor $\tilde{\alpha}_1$ can be associated with the second order derivations of $\hat{f}$. And Thus, it could be considered as an acceleration tensor.

More precisely, let $x \in \mathbb{R}^n$ be the value of a differential application $\phi(\tau)$ depending on a real parameter τ such that $\phi(0) \equiv x_0$, and assuming $\hat{f}(x_0) = x_0 (\equiv 0$ from (1)). Then, denoting by a dot " $\dot{}$ " the derivative with respect to τ, assuming $\hat{x}$ and x are such that $\hat{x} = \hat{f}(x)$ and $\|\dot{x}\|^2 \equiv \omega(\dot{x}\,, \dot{x}) = 1$, and considering a composed application $\hat{g} \circ \hat{f} \in \Gamma_{\widehat{\mathrm{G}}}$ instead of $\hat{f}$ alone, we easily deduce from (15a) the relation:

$$\alpha'(\hat{x}) = \alpha(x) - \ln(\|\dot{\hat{x}}\|) \, .$$

And by differentiating the latter, we obtain also:

$$\alpha'_i(\hat{x}) = \sum_{j=1}^{n} (\hat{f}^{-1})^j_i \circ \hat{f}(x) \;\, \alpha_j(x) - \frac{\ddot{\hat{x}}_i}{\|\dot{\hat{x}}\|^2} \, .$$

In particular, if $\tau = 0$ and $\|\dot{\hat{x}}\|^2 = 1$, then the Jacobian matrice of $\hat{f}$ at x_0 is a Lorentzian matrice and

$$\alpha'(x_0) = \alpha(x_0) \, , \qquad \alpha'_i(x_0) = \sum_{j=1}^{n} (\hat{f}^{-1})^j_i(x_0) \;\, \alpha_j(x_0) - \ddot{\hat{x}}_i \, . \tag{16}$$

But from relations (1), $\hat{f}$ can also be associated to composed diffeomorphisms such as for instance $\varphi^{-1}_{p_0} \circ \varphi'_{p_0}$, where φ'_{p_0} and φ_{p_0} define two different local equivalences at p_0. Then, the points x and $\hat{x}$ can be ascribed, respectively, to moving points p and $\hat{p}$ in the vicinity of p_0. In that case, from the latter geometrical and/or physical interpretation and relations (16), a change of acceleration would keep physical laws invariant since it would be only associated to a change of coordinates. It would be close to Einsteinian relativity and to the Einstein's principle of equivalence. Indeed the "elevator metaphor" in this principle, points out the affine feature of the acceleration which involves equivalence between relative uniformly accelerated frames with relative velocity $\dot{\hat{x}}$ and relative acceleration $\ddot{\hat{x}}$ at their crossing point x_0. This extends the results obtained first by J. Haantjes [29].

Consequently, on the one hand, it is important to notice that μ or equivalently the tensor $\tilde{\alpha}_2 \equiv \{\alpha_{ij}, i, j = 1, \cdots, n\}$ might be considered as an Abraham-Eötvös type tensor [30, 31] encountered in the Eötvös-Dicke experiments for the measurement of the stress-energy tensor of the gravitational potential.

At this point, a first set of physical interpretations (up to a constant for units and with $n = 4$) may be devised: The tensor μ may be ascribed to the stress-energy tensor, $\tilde{\alpha}_1$ to the gravity acceleration 4-vector and α to the Newtonian potential of gravitation.

On the other hand, following W. M. Tulczyjew [32] and J.-F. Pommaret [16], α being associated with dilatations it might be considered as a relative Thomson type temperature: $\alpha = \ln(T_0/T)$, where T_0 is a constant temperature of reference related with the substratum

spacetime $\mathcal{S}$. A first question arises about this temperature T_0: Can we consider it as the 2.7 K cosmic relativistic invariant background temperature ? Also, in view of the transformation law of this function, this would imply that the temperature T is not a conformal invariant but only a Lorentz invariant (as the 2.7 K relic radiation) since in that case $|\det J(\hat{f})| = 1$ and $\ddot{\hat{x}} = 0$ (see also the Helmotz Lorentz invariant representation of the absolute temperature [33] related to a A. H. Taub variational principle [32], and the two opposite non-Lorentz invariant temperature transformation laws given on the one hand by A. Einstein [34], M. von Laue [35], M. Planck [36], and on the other hand by H. Z. Ott [37], H. Arzeliès [38], C. Møller [39]).

In fact, α could be considered either as a Newtonian potential of gravitation or as a temperature (if physically valid), or may be more judiciously as a sum of the two. It appears that physical interpretations could be made at two scales: a "global" one at universe scale ($\mathcal{S}$), and a "local" one ($T_{p_0}\mathcal{M}$) in considering local forces of gravitation attached to a point $p_0 \in \mathcal{M}$. In the same way, a second question arises at the "global" scale about the meaning of the tensor $\tilde{\alpha}_1$ up to units: Since in this case it might be an acceleration (or a gradient of entropy) of reference associated to $\mathcal{S}$, could it be the acceleration of an inflation process (or a time arrow) ? As we shall see, the various dynamics will depend only on the physical interpretations of these two sets of parameters α and $\tilde{\alpha}_1$.

3. The functional dependency of the spacetime deployment

Now we look for the formal series, solutions of the system of PDE (15), assuming from now that the metric ω is analytic. We know these series will be convergent in a suitable open subset and thus providing analytic solutions, since the analytic system is involutive and in particularly elliptic because of a vanishing symbol (see, in appendix 4 of [25], the Malgrange theorem for elliptic systems, given without the assumption of existence of flags of Cauchy data, and generalizing the Cartan-Kähler theorem [40, 41] on analyticity; see also the chapters about the δ-estimate tool for elliptic symbols in [20]; [42]). Nevertheless we need of course to know the Taylor coefficients. For instance we can choose for the applications $\hat{f}$ and the functions α the following series at a point $x_0 \in \mathbb{R}^n$:

$$\hat{f}^i(x) : \ S^i(x, x_0, \{\hat{a}\}) = \sum_{|J|\geq 0}^{+\infty} \hat{a}^i_J (x - x_0)^J / |J|! \,,$$

$$\alpha(x) : \ s(x, x_0, \{c\}) = \sum_{|K|\geq 0}^{+\infty} c_K (x - x_0)^K / |K|! \,,$$

with $x \in U(x_0) \subset \mathbb{R}^n$ being a suitable open neighborhood of x_0 to insure the convergence of the series, $i = 1, \cdots, n$, J and K are multiple index notations such as $J = (j_1, \cdots, j_n)$, $K = (k_1, \cdots, k_n)$ with $|J| = \sum_{i=1}^n j_i$ and similar expressions for $|K|$. Also $\{\hat{a}\}$ and $\{c\}$ are the sets of Taylor coefficients and $\hat{a}^i_J$ and c_K are real values and not functions of x_0, though of course, they can also be values of functions at x_0.

3.1. **The "c system".**

We call the "c system", the system of PDE (15c) (see [43] for an analogous system with $k_0 = 0$). It is from this set of PDE that gauge potentials and fields

of interactions could occur. From the series s, at zero-th order one obtains the algebraic equations ($i,j=1,\cdots,n$; $\boldsymbol{c}_1=\{c_1,\cdots,c_n\}$):

$$c_{ij}=\frac{1}{2}\Big\{k_0(1-\mathrm{e}^{2c_0})-\sum_{k,h=1}^{n}\omega^{kh}(x_0)c_hc_k\Big\}\omega_{ij}(x_0)+c_ic_j+\sum_{k=1}^{n}c_k\gamma_{ij}^k\equiv F_{ij}(x_0,c_0,\boldsymbol{c}_1)\,,\tag{17}$$

and it follows that the c_K's such that $|K|\geq 2$, will depend recursively only on x_0, c_0 and $\boldsymbol{c}_1$. It is none but the least the meaning of formal integrability of so-called involutive systems. Hence the series for α can be written as convergent series $s(x,x_0,c_0,\boldsymbol{c}_1)$ with respect to powers of $(x-x_0)$, c_0 and $\boldsymbol{c}_1$. Let us notice that we can change or not the function α by varying x_0, c_0 or $\boldsymbol{c}_1$.

Let J_1 be the 1-jets affine bundle of the C^∞ real valued functions on $\mathbb{R}^n$. Then it exists a subset associated to $\boldsymbol{c}_0^1\equiv(x_0,c_0,\boldsymbol{c}_1)\in J_1$, we denote by $\mathcal{S}_c^1(\boldsymbol{c}_0^1)\subset J_1$, the set of elements $(x_0',c_0',\boldsymbol{c}_1')\in J_1$, such that there is an open neighborhood $U(\boldsymbol{c}_0^1)\subset\mathcal{S}_c^1(\boldsymbol{c}_0^1)$, projecting on $\mathbb{R}^n$ in an open neighborhood of a given $x\in U(x_0)$, for which for all $(x_0',c_0',\boldsymbol{c}_1')\in U(\boldsymbol{c}_0^1)$ then $s(x,x_0,c_0,\boldsymbol{c}_1)=s(x,x_0',c_0',\boldsymbol{c}_1')$. Assuming the variation ds with respect to x_0, c_0 and $\boldsymbol{c}_1$ is vanishing, at a given fixed x, is the subset $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ a submanifold of J_1 ? From $ds\equiv 0$ it follows that ($k=1,\cdots,n$):

$$\sigma_0\equiv dc_0-\sum_{i=1}^{n}c_i\,dx_0^i=0\,,\qquad \sigma_k\equiv dc_k-\sum_{j=1}^{n}F_{kj}(x_0,c_0,\boldsymbol{c}_1)\,dx_0^j=0\,.$$

We recognize a regular analytic Pfaff system we denote P_c, generated by the 1-forms σ_0 and σ_k, and the meaning of their vanishing is that the solutions α do not change for such variations of c_0, $\boldsymbol{c}_1$ and x_0. Also, as can be easily verified, the Pfaff system P_c is integrable since the Fröbenius condition of involution is satisfied, and all the prolongated 1-forms σ_K ($|K|\geq 2$) will be linear combinations of these $n+1$ generating forms thanks to the recursion property of formal integrability. Then the subset $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ of dimension n containing a particular element $\boldsymbol{c}_0^1\equiv(x_0,c_0,\boldsymbol{c}_1)$ is a submanifold of J_1. It is a particular leaf of, at least, a local foliation on J_1 of codimension $n+1$.

Since the system of PDE defined by the involutive Pfaff system P_c, namely the c system, is elliptic (i.e. vanishing symbol) and formally integrable, one deduces that it exists on J_1, local systems of coordinates $(x_0,\tau_0,\tau_1,\cdots,\tau_n)$ such that each leaf $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ is an analytic submanifold [25] for which $\tau_0=cst$ and $\tau_i=cst$ ($i=1,\cdots,n$). This involves that all the convergent series $s(x,x_0',c_0',\boldsymbol{c}_1')$ with $(x_0',c_0',\boldsymbol{c}_1')\in\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ equal one only analytic solution function $u(x,\tau_0,\boldsymbol{\tau}_1)$ ($\boldsymbol{\tau}_1\equiv\{\tau_1,\cdots,\tau_n\}$), analytic with respect to x as well as with respect to the τ's. This results of the continuous series s which are convergent whatever the fixed set of given values x, x_0, c_0 and $\boldsymbol{c}_1$. Thus, in full generality, considering the difference $s(x,x_0,c_0,\boldsymbol{c}_1)-s(x,x_0',c_0',\boldsymbol{c}_1')$ we have the relation:

$$s(x,x_0,c_0,\boldsymbol{c}_1)-s(x,x_0',c_0',\boldsymbol{c}_1')=u(x,\tau_0,\boldsymbol{\tau}_1)-u(x,\tau_0',\boldsymbol{\tau}_1')\,,\tag{18}$$

with τ parameters related by ($i=1,\cdots,n$)

$$\triangle_0\tau\equiv\tau_0'-\tau_0=\int_{\boldsymbol{c}_0^1}^{\boldsymbol{c}'^1_0}\sigma_0\,,\qquad \triangle_i\tau\equiv\tau_i'-\tau_i=\int_{\boldsymbol{c}_0^1}^{\boldsymbol{c}'^1_0}\sigma_i\,.\tag{19}$$

Now, we consider the c's are values of differential (i.e. C^∞) functions ρ: $c_K = \rho_K(x_0)$, as expected for Taylor series coefficients, and defined on a starlike open neighborhood of x_0. Roughly speaking, we make a pull-back on $\mathbb{R}^n$ by differentiable sections, inducing a projection from the subbundle of projectable elements in T^*J_1 to $T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_1$. Then, we set (with $\boldsymbol{\rho}_1 \equiv \{\rho_1, \cdots, \rho_n\}$ and no changes of notations for the pull-backs):

$$\sigma_0 \equiv \sum_{i=1}^{n} (\partial_i \rho_0 - \rho_i)\, dx_0^i \equiv \sum_{i=1}^{n} \mathcal{A}_i\, dx_0^i\,, \tag{20a}$$

$$\sigma_i \equiv \sum_{j=1}^{n} (\partial_j \rho_i - F_{ij}(x_0, \rho_0, \boldsymbol{\rho}_1))\, dx_0^j \equiv \sum_{j=1}^{n} \mathcal{B}_{j,i}\, dx_0^j\,, \tag{20b}$$

and it follows the integrals (19) must be performed from x_0 to x_0' in a starlike open neighborhood of x_0. In particular, if $\boldsymbol{c}_0^1$ is an element of the "null" submanifold corresponding to the vanishing solution of the "c system", then the difference (18) involves that

$$\alpha(x) \equiv s(x, x_0', c_0', \boldsymbol{c}_1') = u(x, \tau_0', \boldsymbol{\tau}_1')\,,$$

with

$$\tau_0' = \int_{x_0}^{x_0'} \sum_{i=1}^{n} \mathcal{A}_i\, dx^i + \tau_0\,, \qquad \tau_i' = \int_{x_0}^{x_0'} \sum_{j=1}^{n} \mathcal{B}_{j,i}\, dx^j + \tau_i\,.$$

Then we deduce:

Theorem 1. *All the analytic solutions of the involutive system of PDE* (15c) *can be written in a suitable starlike open neigborhood of* x_0 *as*

$$\alpha(x) \equiv u\Big(x, \int_{x_0}^{x_0'} \sum_{i=1}^{n} \mathcal{A}_i\, dx'^i + \tau_0\,, \int_{x_0}^{x_0'} \sum_{j=1}^{n} \mathcal{B}_{j,1}\, dx'^j + \tau_1, \cdots, \int_{x_0}^{x_0'} \sum_{j=1}^{n} \mathcal{B}_{j,n}\, dx'^j + \tau_n\Big)\,, \tag{21}$$

with $u(x_0, \tau_0, \tau_1, \cdots, \tau_n) = 0$, $x_0' \in U(x_0)$ *and where* u *is a unique fixed analytic function depending on the n(n+1)* C^∞ *functions* $\mathcal{A}_i$ *and* $\mathcal{B}_{j,k}$ *defined by the relations* (20). *The integrals in* u *are called the "potential of interactions". Let us remark that we can set* $x_0' \equiv v(x)$ *if the gradient of* v*, i.e.* ∇v*, is in the annihilator of the Pfaff system* P_c *of 1-forms* σ*.*

Also this result shows the functional dependencies of the solutions of the "c system" with respect to the functions ρ_0 and $\boldsymbol{\rho}_1$, themselves associated to the smooth infinitesimal deformations of these solutions. These smooth infinitesimal deformations gauge fields $\mathcal{A}$ and $\mathcal{B}$, defined by $n(n+1)$ potential functions (**20** functions if $n = 4$), can also be considered as infinitesimal smooth deformations from "Poincaré solutions" of the system (15) for which $\alpha \equiv 0$, to some "conformal solutions" whatever is α.

Moreover the functions ρ, and consequently the functions ρ_0, $\mathcal{A}$ and $\mathcal{B}$, must satisfy additional differential equations coming from Fröbenius conditions of involution of the Pfaff system P_c. More precisely, from the relations $d\sigma_0 = \sum_{i=1}^n dx_0^i \wedge \sigma_i$, $d\sigma_i = \sum_{j=1}^n dx_0^j \wedge \sigma_{ij}$ and

$$\sigma_{ij} = c_i \sigma_j + c_j \sigma_i - \omega_{ij}\Big\{k_0 \mathrm{e}^{2c_0} \sigma_0 + \sum_{k,h=1}^{n} \omega^{kh} c_h \sigma_k\Big\} + \sum_{k=1}^{n} \gamma_{ij}^k \sigma_k \equiv \vartheta_{ij}(\boldsymbol{c}_0^1, \sigma_J; |J| \leq 1)\,, \tag{22}$$

one deduces a set of algebraic relations at x_0:

$$\mathcal{J}_{k,j,i} \equiv \omega_{ij} \left\{ k_0 e^{2\rho_0} \mathcal{A}_k + \sum_{r,s=1}^{n} \omega^{rs} \rho_r \mathcal{B}_{k,s} \right\} + \partial_j \mathcal{B}_{k,i} - \rho_i \mathcal{B}_{k,j} - \rho_j \mathcal{B}_{k,i} - \sum_{s=1}^{n} \gamma_{ij}^{s} \mathcal{B}_{k,s} \,, \tag{23a}$$

$$\mathcal{I}_{i,k} = \mathcal{I}_{k,i} \equiv \partial_k \mathcal{A}_i - \mathcal{B}_{i,k} \,, \qquad \mathcal{J}_{k,j,i} = \mathcal{J}_{j,k,i} \,. \tag{23b}$$

Clearly, in these relations, the set of functions $(\rho_0, \boldsymbol{\rho}_1)$ appears to be a set of arbitrary differential functions. In considering $\mathcal{F}$ and $\mathcal{G}$ as being respectively the skew-symmetric and the symmetric parts of the tensor of components $\partial_i \rho_j$, then one deduces, from the symmetry properties of the latter relations, what we call *the first set of differential equations*:

$$\partial_i \mathcal{F}_{jk} + \partial_j \mathcal{F}_{ki} + \partial_k \mathcal{F}_{ij} = 0 \,, \tag{24a}$$

$$2\, \partial_j \mathcal{G}_{ki} - \partial_i \mathcal{G}_{kj} - \partial_k \mathcal{G}_{ij} = \partial_i \mathcal{F}_{jk} - \partial_k \mathcal{F}_{ij} \,, \tag{24b}$$

with

$$\mathcal{F}_{ij} = \partial_j \rho_i - \partial_i \rho_j = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \,, \tag{25a}$$

$$\mathcal{G}_{ij} = -(\partial_i \rho_j + \partial_j \rho_i) \equiv \partial_i \mathcal{A}_j + \partial_j \mathcal{A}_i \mod (\rho_0, \boldsymbol{\rho}_1) \,. \tag{25b}$$

The PDE (24a) with (25a) might be interpreted as the first set of *Maxwell equations*. In view of physical interpretations, we can easily compute the Euler-Lagrange equations of a conformally equivariant Lagrangian density

$$\mathcal{L}(x_0, \rho_0, \boldsymbol{\rho}_1, \mathcal{A}, \mathcal{F}, \mathcal{G})\, d^n x_0 \,, \tag{26}$$

with $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$ satisfying the relations (20) and (25). We would obtain easily what we call *the second set of differential equations.*

Then we give a few definitions to proceed further.

Definition 1. We denote:

(1) $\theta_{\mathbb{R}}$, the presheaf of rings of germs of the differential (i.e. C^∞) functions defined on $\mathbb{R}^n$,
(2) $\underline{J_1}$, the presheaf of $\theta_{\mathbb{R}}$-modules of germs of differential sections of J_1,
(3) $\underline{\mathcal{S}_c^0} \subset \theta_{\mathbb{R}}$, the presheaf of rings of germs of functions which are solutions with their first derivatives, of the "algebraic equations" GHSS (15c) taken at any given points x_0 in $\mathbb{R}^n$, <u>not simultaneously</u> at each point in $\mathbb{R}^n$ (see *Remark 1* below),
(4) $\mathcal{S}_c^1 \subset \underline{J_2}$, projectable on $\underline{J_1}$ ($\underline{J_1} \simeq \mathcal{S}_c^1$), the embedding in $\underline{J_2}$ of the presheaf of $\theta_{\mathbb{R}}$-modules of germs of differential sections of J_2, defined by the system (15c) of algebraic equations at any given points $x_0 \in \mathbb{R}^n$ (not everywhere, as mentioned above),
(5) $\underline{T^*\mathbb{R}^n}$, the presheaf of $\theta_{\mathbb{R}}$-modules of germs of global 1-forms on $\mathbb{R}^n$.

Remark 1: Through this set of definitions, we do not consider PDEs solutions, but instead, solutions of algebraic equations at any given point x_0. In this light, PDEs solutions are to be regarded as particular "coherent" subsheafs for which equations (15c) are satisfied everywhere in $\mathbb{R}^n$, i.e., at $x \neq x_0$, and not solely at x_0. We insist that the algebraic equations (15c) do not concern solutions of a PDE system, but the values of

second derivatives of functions at x_0, depending on those of first order at most at x_0, with no constraints between first and zero-th order values of these functions at x_0.

Then, considering the local diffeomorphisms

$$(\wedge^k T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_r)_{x_0} \simeq (\{x_0\} \otimes_{\mathbb{R}} J_r) \times (\wedge^k T^*_{x_0}\mathbb{R}^n \otimes_{\mathbb{R}} J_r)$$

with $0 \leq k \leq n$ and $r \geq 0$, we set the definitions:

Definition 2. We define the local operators:

(1) $j_1 : (x_0, \rho_0) \in \mathcal{S}^0_c \longrightarrow (x_0, \rho_0, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathcal{S}^1_c$ with $\boldsymbol{\rho}_1 = (\partial_1\rho_0, \cdots, \partial_n\rho_0)$ and $\boldsymbol{\rho}_2 = (\partial^2_{11}\rho_0, \partial^2_{12}\rho_0, \cdots, \partial^2_{nn}\rho_0)$,

(2) $D_{1,c} : \boldsymbol{\rho}^2_0 \equiv (x_0, \rho_0, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathcal{S}^1_c \longrightarrow (\boldsymbol{\rho}^1_0, \sigma_0, \sigma_1, \cdots, \sigma_n) \in \underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}$, with $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{\rho}^1_0 \equiv (x_0, \rho_0, \boldsymbol{\rho}_1)$ satisfying relations (20), and $P_c = \{\sigma_0, \sigma_1, \cdots, \sigma_n\}$ being a Pfaffian system of linearly independent regular 1-forms on J_1,

(3) $D_{2,c} : (\boldsymbol{\rho}^1_0, \sigma_0, \sigma_1, \cdots, \sigma_n) \in \underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1} \longrightarrow (\boldsymbol{\rho}^1_0, \zeta_0, \zeta_1, \cdots, \zeta_n) \in \wedge^2\underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}$, with

$$\zeta_0 = \sum_{i,j=1}^{n} \mathcal{I}_{i,j}\, dx^i_0 \wedge dx^j_0\,, \qquad \zeta_k = \sum_{i,j=1}^{n} \mathcal{J}_{j,i,k}\, dx^i_0 \wedge dx^j_0\,,$$

the functions $(x_0, \rho_0, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathcal{S}^1_c$ and the tensors $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ satisfying the relations (23).

Then from all that preceeds, we can deduce:

Theorem 2. *The differential sequence*

$$0 \longrightarrow \mathcal{S}^0_c \xrightarrow{j_2} \mathcal{S}^1_c \xrightarrow{D_{1,c}} \underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1} \xrightarrow{D_{2,c}} \wedge^2\underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}\,,$$

with the $\mathbb{R}$-linear local differential operators $D_{1,c}$ and $D_{2,c}$, is exact (where the first injectivity, namely j_1, results from remark 1).

Remark 2: Before proceeding with the proof of this Theorem, a few comments are in order. The continuation "on the right" of the differential sequence above would require, in order to demonstrate the exactness, a generalization of the Fröbenius Therorem to p-forms with $p \geq 2$, which, to our knowledge at least, is not available in full generality, no more as the concept of canonical contact p-forms representations. Indeed, the higher local differential operators $D_{i\geq 3,c}$ would be non-linear, in contrary to the usual Spencer differential operators, because of the non-linearity of the GHSS system. We are faced to the same situation encountered in the Spencer sequences for Lie equations, these sequences being truncated at this same order two. The sequence above is a physical gauge sequence, for which we can make the following identification: $\underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}$ is the space of the gauge *potentials* $\mathcal{A}$ and $\mathcal{B}$, whereas $\wedge^2\underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}$ is the space of the gauge *strength fields* $\mathcal{I}$ and $\mathcal{J}$.

This sequence is close to a kind of Spencer linear sequence [20]. It differs essentially in the tensorial product which is taken on $\theta_{\mathbb{R}}$ (because of the non-linearity of the "GHSS system", inducing a ρ "dependence" of the various Pfaff forms) rather than on the $\mathbb{R}$ field as is in the original linear Spencer theory [20] (other developements have included the $\theta_{\mathbb{R}}$ case after this first Spencer original version). Also, since the system P_c is integrable, it is

always, at least locally, diffeomorphic to an integrable set of Cartan 1-forms in $T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_1$ associated to a particular finite Lie algebra g_c (of dimension greater or equal to $n+1$), with corresponding Lie group G_c acting on the left on each leaf of the foliation $\mathcal{F}_1$ [45, 44]. It follows that the integrals in (21) would define a deformation class in the first non-linear Spencer cohomology space of deformations of global sections from $\mathbb{R}^n$ to a sheaf of Lie groups G_c [26] (see also [46], though within a different approach).

In addition, 1) in Theorem 1, the fonction u is defined with integrals associated to the definition of a homotopy operator of the differential sequence above [**?**], and 2) in Theorem 2, the metric ω is allowed to be of class C^∞, rather than analytic, as in Theorem 1, because formal properties only are considered.

Proof of Theorem 2: At $\mathcal{S}_c^1$ the sequence exactness is trivial and we may pass to the exactness of the differential sequence at $\underline{T^*\mathbb{R}^n} \otimes_{\theta_{\mathbb{R}}} \underline{J_1}$.

In a neighbourhood of an open set $V(\boldsymbol{C}_0^1) \subset J_1$ of $\boldsymbol{C}_0^1 \in J_1$, the condition $D_{2,c}(\sigma) = 0$ implies the relations:

$$d\tilde{\sigma}_0 = \textstyle\sum_{i=1}^n dx_0^i \wedge \tilde{\sigma}_i \,, \tag{27}$$

$$d\tilde{\sigma}_i = \textstyle\sum_{j=1}^n dx_0^j \wedge \tilde{\sigma}_{ij} \tag{28}$$

with:

$$\tilde{\sigma}_{ij} = c_i\tilde{\sigma}_j + c_j\tilde{\sigma}_i - \omega_{ij}\Big\{k_0 e^{2c_0}\tilde{\sigma}_0 + \sum_{k,h=1}^n \omega^{kh} c_h \tilde{\sigma}_k\Big\} + \sum_{k=1}^n \gamma_{ij}^k \tilde{\sigma}_k \,, \tag{29}$$

the "$\tilde{\sigma}$" 1-forms are defined above J_1, and correspond to the 1-forms σ defined in a neighbourhood $W(X_0) \subset \mathbb{R}^n$ at $x_0 \in W(X_0)$: they are such that if $p_1 : J_1 \longrightarrow \mathbb{R}^n$ stands for the standard projection, then $p_1(V(\boldsymbol{C}_0^1)) = W(X_0)$, $p_1(\boldsymbol{c}_0^1) = x_0$ and $p_1(\boldsymbol{C}_0^1) = X_0$.

Regularity and linear independence, ensure the existence of a locally integrable manifold $\mathcal{V}_1$, with dimension n, and of $n+1$ first integrals $\{y_\nu\}$ ($\nu = 0, 1, \ldots, n$). Up to constants, the functions y_ν can be choosen such that $y_\nu(\boldsymbol{C}_0^1) = 0$. Then, at $\boldsymbol{C}_0^1$, we have the relations:

$$\tilde{\sigma}_\nu(\boldsymbol{C}_0^1) \equiv dy_\nu/_{\boldsymbol{C}_0^1} \,, \tag{30}$$

and in $V(\boldsymbol{C}_0^1)$, the relations

$$\tilde{\sigma}_\nu = dy_\nu - \sum_{\mu\neq\nu}^n f_\nu^\mu(y) dy_\mu \,, \tag{31}$$

with $f_\nu^\mu(y) \longrightarrow 0$ when $y \longrightarrow 0$, that is when $\boldsymbol{c}_0^1 \longrightarrow \boldsymbol{C}_0^1$.

These relations can also be defined on the presheaves of the J_1 local sections. This is because it exists a C^1-mapping, say s, from $W(X_0)$ into $V(\boldsymbol{C}_0^1)$, such that $s(W(X_0)) = U(\boldsymbol{C}_0^1) \subset V(\boldsymbol{C}_0^1)$ and $s(x_0) = \boldsymbol{c}_0^1$. And thus locally, one has $\mathcal{V}_1 \cap U(\boldsymbol{C}_0^1) \simeq W(X_0)$. In the relations (31), it is therefore possible to take $y_j \equiv c_j$ ($j = 1, \ldots, n$). Setting $s^*(dy_j) = s^*(dc_j) \equiv dx_0^j$, and denoting by "$\rho$" the functions $\rho_j(x_0) = c_j$ and $\rho_0(x_0) = y_0$, associated to s, we have immediately in particular $\sigma_0 = s^*(\tilde{\sigma}_0)$, and $\forall\, x_0$:

$$\sigma_0 \equiv d\rho_0 - \sum_{i=0}^n f_0^i(\rho)\, dx_0^i \,. \tag{32}$$

We set $\rho_i \equiv f_0^i(\rho)$. Now, from (32) and the pull-back of (27), one deduces that

$$\sum_{i=1}^{n} dx_0^i \wedge (d\rho_i - \sigma_i) = 0\,, \tag{33}$$

and in particular:

$$dx_0^1 \wedge dx_0^2 \wedge \cdots \wedge dx_0^n \wedge (d\rho_i - \sigma_i) = 0\,. \tag{34}$$

Consequently

$$d\rho_i - \sigma_i = \sum_{j=1}^{n} \rho_{i,j}\, dx_0^j \Longleftrightarrow \sigma_i = d\rho_i - \sum_{j=1}^{n} \rho_{i,j}\, dx_0^j\,, \tag{35}$$

that are alternatives to the pull-backs by s of relations (31). We thus have,

$$\sigma_0 = d\rho_0 - \sum_{i=1}^{n} \rho_i\, dx_0^i\,, \tag{36}$$

$$\sigma_i = d\rho_i - \sum_{j=1}^{n} \rho_{i,j}\, dx_0^j\,. \tag{37}$$

Now, out of (37) and the pull-backs of (28), one deduces also the relations,

$$\sum_{j=1}^{n} dx_0^j \wedge d\rho_{i,j} = \sum_{j=1}^{n} dx_0^j \wedge \sigma_{ij} \Longleftrightarrow \sum_{j=1}^{n} dx_0^j \wedge (d\rho_{i,j} - \sigma_{ij}) = 0\,. \tag{38}$$

By the same procedure as above, we thus get:

$$d\rho_{i,j} - \sigma_{ij} = \sum_{j=1}^{n} \rho_{i,j,k}\, dx_0^k \Longleftrightarrow \sigma_{ij} = d\rho_{i,j} - \sum_{k=1}^{n} \rho_{i,j,k}\, dx_0^k\,. \tag{39}$$

Moreover, in view of (33) and (37) we deduce the symmetries: $\rho_{i,j} = \rho_{j,i} \equiv \rho_{ij}$ and $\rho_{i,j,k} = \rho_{j,i,k} \equiv \rho_{ij,k}$. Then, considering the coefficients of a same basis differential form with $\boldsymbol{\rho}_2 \equiv (\rho_{ij})$, $\boldsymbol{\rho}_1 \equiv (\rho_i)$, and the system of algebraic equations for $\boldsymbol{\rho}_0^2$ deduced from (27) and (28), we conclude that $\boldsymbol{\rho}_0^2 \in \mathcal{S}_c^1$. $\square$

In order to know the effects on $\mathcal{M}$ of these infinitesimal deformations, we need to describe what are their incidences upon the objects acting primarily on $\mathbb{R}^n$, namely the applications $\hat{f}$. Thus, we pass to the study of what we call the "ab system" of the PDE system (15).

3.2. **The "ab system".** This system is defined by the first two sets of PDE (15a) and (15b). For this system of Lie equations, we will begin with recalling well-known results but in the framework of the present context. Applying the same reasoning than in the

previous subsection, we first obtain the following results, which hold up to order two:

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{a}_0)\ \hat{a}^r_i\ \hat{a}^s_j = e^{2c_0}\omega_{ij}(x_0)\,, \tag{40a}$$

$$\hat{a}^k_{ij} + \sum_{r,s=1}^{n} \gamma^k_{rs}(\hat{a}_0)\ \hat{a}^r_i\ \hat{a}^s_j = \sum_{q=1}^{n} \hat{a}^k_q \left(\gamma^q_{ij}(x_0) + c_i\delta^q_j + c_j\delta^q_i - \omega_{ij}(x_0)c^q\right)\,, \tag{40b}$$

which clearly show that $J_1(\mathbb{R}^n)$ is diffeomorphic to an embeded submanifold of the 2-jets affine bundle $J_2(\mathbb{R}^n)$ of the $C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ differentiable applications on $\mathbb{R}^n$. In second place, we get relations, from the (T) system, for the coefficients of order 3 that we only write as ($\hat{a}_1 \equiv (\hat{a}^i_j)$; $\hat{a}_2 \equiv (\hat{a}^i_{jk})$,..., $\hat{a}_k \equiv (\hat{a}^i_{j_1\cdots j_k})$; $\hat{\boldsymbol{a}}^k_0 \equiv (\hat{a}_0,\cdots,\hat{a}_k)$):

$$\hat{a}^i_{jkh} \equiv \hat{A}^i_{jkh}(x_0,\hat{\boldsymbol{a}}^2_0)\,, \tag{41}$$

where $\hat{A}^i_{jkh}$ are algebraic functions, pointing out in this expression the independency from the "c" coefficients (also like the relations (40a), for instance, when expressing c_0 with respect to the determinant of $\hat{a}_1$). We denote by $\widehat{\Omega}^i_J$ the Pfaff 1-forms at x_0 and $\{\hat{a}\}$ (or at $(x_0,\{\hat{a}\})$):

$$\widehat{\Omega}^i_J \equiv d\hat{a}^i_J - \sum_{k=1}^{n} \hat{a}^i_{J+1_k} dx^k_0\,, \tag{42}$$

and setting the $\hat{a}$'s as values of functions $\hat{\tau}$ depending on x_0 (in some way we make a pull-back on $\mathbb{R}^n$), we define the tensors $\hat{\kappa}$ by:

$$\widehat{\Omega}^i_J \equiv \sum_{k=1}^{n}\left(\partial_k\hat{\tau}^i_J - \hat{\tau}^i_{J+1_k}\right)dx^k_0 \equiv \sum_{k=1}^{n}\hat{\kappa}^i_{k,J}\,dx^k_0\,. \tag{43}$$

Then from the relations:

$$\mathrm{e}^{2c_0}\omega^{rs}(\hat{a}_0) = \sum_{i,j=1}^{n}\omega^{ij}(x_0)\hat{a}^r_i\hat{a}^s_j\,, \qquad \sum_{i=1}^{n}\gamma^i_{ik} = \frac{1}{2}\sum_{i,j=1}^{n}\omega^{ij}\,\partial_k\omega_{ij}\,,$$

we deduce with $\hat{b} \equiv \hat{a}_1^{-1}$, for example, that the $\widehat{\Omega}^i_j$ 1-forms satisfy the relations at $(x_0,\hat{\boldsymbol{a}}^1_0)$:

$$\widehat{H}_0(x_0,\hat{\boldsymbol{a}}^1_0,\widehat{\Omega}^k_L;|L|\leq 1) \equiv \sum_{i,j=1}^{n}\hat{b}^j_i\,\widehat{\Omega}^i_j + \sum_{j,k=1}^{n}\gamma^j_{jk}(\hat{a}_0)\widehat{\Omega}^k = n\sigma_0\,. \tag{44}$$

Similar computations show that the 1-forms σ_i can be expressed as quite long relations, linear in the $\widehat{\Omega}^j_J$ ($|J|\leq 2$), with coefficients which are algebraic functions depending on the $\hat{a}_K$ ($|K|\leq 2$), the derivatives of the metric and the Riemann-Christoffel symbols, all of them taken either at x_0 or $\hat{a}_0$. Then, we set:

$$\sigma_i \equiv \widehat{H}_i(x_0,\hat{\boldsymbol{a}}^2_0,\widehat{\Omega}^j_I;|I|\leq 2)\,. \tag{45}$$

From (41) the 1-forms $\widehat{\Omega}^i_{jkh}$ are also sums of 1-forms $\widehat{\Omega}^r_K$ ($|K|\leq 2$) with the same kind of coefficients and not depending on the σ's, and we write:

$$\widehat{\Omega}^i_{jkh} \equiv \widehat{K}^i_{jkh}(x_0,\hat{\boldsymbol{a}}^2_0,\widehat{\Omega}^r_K;|K|\leq 2)\,, \tag{46}$$

where $\widehat{K}^i_{jkh}$ are functions which are linear in the 1-forms $\widehat{\Omega}^r_K$.

Let us denote by $\widehat{\boldsymbol{\mathcal{P}}}_2 \subset J_2(\mathbb{R}^n)$ the set of elements $(x_0, \hat{\boldsymbol{a}}^2_0)$ satisfying relations (40) whatever are the c's. Then the Pfaff system we denote $\widehat{P}_2$ over $\widehat{\boldsymbol{\mathcal{P}}}_2$ and generated by the 1-forms $\widehat{\Omega}^j_K \in T^*\mathbb{R}^n \otimes_{\mathbb{R}} J_2(\mathbb{R}^n)$ in (42) with $|K| \leq 2$, is locally integrable on every neighborhood $U(x_0, \hat{\boldsymbol{a}}^2_0) \subset J_2(\mathbb{R}^n)$, since at $(x_0, \hat{\boldsymbol{a}}^2_0)$ we have ($|J| \leq 2$):

$$d\widehat{\Omega}^i_J - \sum_{k=1}^n dx^k_0 \wedge \widehat{\Omega}^i_{J+1_k} \equiv 0\,, \tag{47}$$

together with (46).

From now on, we consider the "Poincaré system" whose corresponding notations will be free of "hats". We denote by Ω^i_J the Pfaff 1-forms corresponding to this system, i.e. the system defined by the PDE (15a) and (15b) with a vanishing function α. The corresponding 1-forms "σ" are also vanishing everywhere on $\mathbb{R}^n$ and the Ω^i_J satisfy all of the previous relations with the σ's cancelled out. Then it is easy to see the Ω^i_J 1-forms ($|J| \geq 2$) are generated by the set of 1-forms Ω^j_K ($|K| \leq 1$), and in particular we have

$$\Omega^k_{ij} = -\left\{ \sum_{r,s,h=1}^n (\partial_h \gamma^k_{rs})(a_0)\Omega^h\, a^r_i\, a^s_j + \sum_{r,s=1}^n \gamma^k_{rs}(a_0)[\, a^r_i\, \Omega^s_j + a^s_j\, \Omega^r_i\,] \right\}\,, \tag{48}$$

with $(x_0, \hat{\boldsymbol{a}}^1_0 \equiv \boldsymbol{a}^1_0) \in \boldsymbol{\mathcal{P}}_1 \subset J_1(\mathbb{R}^n)$, and $\boldsymbol{\mathcal{P}}_1$ being the set of elements satisfying relations (40a) with $c_0 = 0$. Similarily the Pfaff system we denote P_1 over $\boldsymbol{\mathcal{P}}_1$ and generated by the 1-forms Ω^j_K in (42) with $|K| \leq 1$, is locally integrable on every neighborhood $U(x_0, \boldsymbol{a}^1_0) \subset \boldsymbol{\mathcal{P}}_1$, since at the point $(x_0, \boldsymbol{a}^1_0)$ we have relations (47) with $|J| \leq 1$ together with relations (48).

Then at each $(x_0, \hat{\boldsymbol{a}}^2_0) \in \boldsymbol{\mathcal{P}}_2$, we have the locally exact splitted sequence

$$0 \longrightarrow P_1 \xrightarrow{b_1} \widehat{P}_2 \xrightarrow{e_1} P_c \longrightarrow 0\,, \tag{49}$$

where we consider $J_1(\mathbb{R}^n)$ embeded in $J_2(\mathbb{R}^n)$ as well as $\boldsymbol{\mathcal{P}}_1$ in $\widehat{\boldsymbol{\mathcal{P}}}_2 \supset \boldsymbol{\mathcal{P}}_2$ from relations (40). In this sequence a back connection b_1 and a connection $c_1 : P_c \longrightarrow \widehat{P}_2$ are such that ($|J| \leq 2$):

$$\widehat{\Omega}^i_J = \Omega^i_J + \chi^i_J(x_0, \hat{\boldsymbol{a}}^2_0)\, \sigma_0 + \sum_{k=1}^n \chi^{i,k}_J(x_0, \hat{\boldsymbol{a}}^2_0)\, \sigma_k\,, \tag{50}$$

with Ω^i_{jk} satisfying (48) for any given Ω^h_L with $|L| \leq 1$, and where the tensors χ are defined on $\boldsymbol{\mathcal{P}}_2$. Together, they define a back connection, and the tensors χ define a connection if they satisfy the relations:

$$\widehat{H}_0(x_0, \hat{\boldsymbol{a}}^1_0, \chi^k_L; |L| \leq 1) = n\,, \qquad \widehat{H}_0(x_0, \hat{\boldsymbol{a}}^1_0, \chi^{k,i}_L; |L| \leq 1) = 0\,,$$
$$\widehat{H}_i(x_0, \hat{\boldsymbol{a}}^2_0, \chi^k_L; |L| \leq 2) = 0\,, \qquad \widehat{H}_i(x_0, \hat{\boldsymbol{a}}^2_0, \chi^{k,h}_L; |L| \leq 2) = n\delta^h_i\,,$$

in order to preserve relations (44) and (45), i.e. $e_1 \circ c_1 = id$.

4. The spacetime $\mathcal{M}$ unfolded by gravitation and electromagnetism

Again, in view of physical interpretations, we put a spotlight on the tensor $\mathcal{B}$. In fact, we consider the relations (50) with $|J| = 0$ and the $\widehat{\Omega}^i$ as fields of *"tetrads"*. Then we get a metric ν for the *"unfolded spacetime $\mathcal{M}$"* defined infinitesimally at x_0 (corresponding to p_0 in $\mathcal{M}$) by:

$$\nu = \tilde{\omega} = \omega + \delta\omega = \sum_{i,j=1}^{n} \omega_{ij} \circ \hat{\tau}(x_0)\, \widehat{\Omega}^i(x_0) \otimes \widehat{\Omega}^j(x_0)\,,$$

$$\widehat{\Omega}^i = \sum_{k=1}^{n} \hat{\kappa}^i_k(x_0)\, dx_0^k\,,$$

$$\hat{\kappa}^i_j(x_0) = \kappa^i_j(x_0) + \chi^i(x_0, \boldsymbol{\tau}_0^2)\, \mathcal{A}_j(x_0) + \sum_{k=1}^{n} \chi^{i,k}(x_0, \boldsymbol{\tau}_0^2)\, \mathcal{B}_{j,k}(x_0)\,.$$

We consider the particular case for which the S-admissible metric ω is equal to $\mathrm{diag}[+1, -1, \cdots, -1]$ (and thus $k_0 = 0$), the χ's are only depending on x_0 and $\kappa^i_j = \delta^i_j$, i.e. the deformation of ω is only due to the tensors $\mathcal{A}$ and $\mathcal{B}$. Thus, one has the general relation between ν and ω: $\nu = \omega +$ linear and quadratic terms in $\mathcal{A}$ and $\mathcal{B}$. Then from this metric ν, one can deduce the Riemann and Weyl curvature tensors of the *"unfolded spacetime $\mathcal{M}$"*.

We are faced with a question: could this kind of deformation be interpreted as an inflation process in cosmology (due to a kind of instable substratum spacetime $\mathcal{S}$ for instance) ? In such a process, each occurrence of a creation or annihilation of singularities of the gauge potentials $\mathcal{A}$ and $\mathcal{B}$, would be related to a non-trivial unfolding, i.e. a non-smooth deformation.

Also, would the inflation be the evolution from the Poincaré Lie structure to "a" conformal one, and going from a physically "homogeneous" spacetime (namely with constant Riemann curvature and a vanishing Weyl tensor and so *"conformally flat"*) to an "inhomogeneous" one (with any Weyl tensor) ? Or, would the substratum spacetime $\mathcal{S}$ be merely a kind of "mean" smooth manifold $\mathcal{M}$ with no inflation process ?

In some way, the singularities of the gauge potentials $\mathcal{A}$ and $\mathcal{B}$ would produce a kind of *"bifurcation"* of the unfolded spacetime structure leading to a different concept of bifurcation than the one used in the case of non-linear ODE's.

In view of making easier computations for a relativistic action deduced from the metric tensor ν, we consider this metric in the *"weak fields limit"*, assuming that the metric ν is linear in the tensors $\mathcal{A}$ and $\mathcal{B}$ and that the quadratic terms can be neglected. Furthermore, from relations (23), we have the relations:

$$\partial_i \mathcal{A}_k - \partial_k \mathcal{A}_i = \mathcal{B}_{k,i} - \mathcal{B}_{i,k} = \mathcal{F}_{ik}\,, \qquad \partial_j \mathcal{B}_{k,i} - \partial_k \mathcal{B}_{j,i} \simeq 0\,,$$

since the functions ρ take also small values in the weak fields limit. Therefore, we can write $\nu_{ij} \simeq \omega_{ij} + \epsilon_{ij}$, where the coefficients ϵ_{ij} can be viewed as small perturbations of the metric ω and linearly defined from $\mathcal{A}$, $\mathcal{B}$ and the χ tensors. Then, let i be a differential map $i : s \in [0, \ell] \subset \mathbb{R} \longrightarrow i(s) = x_0 \in \mathbb{R}^n \simeq \mathcal{M}$, and U being such that $U \equiv di(s)/ds$,

$\|U\|^2 = 1$. We define the relativistic action S_1 by:

$$S_1 = \int_0^\ell \sqrt{\nu(U(s), U(s))}\, ds \equiv \int_0^\ell \sqrt{L_\nu}\, ds\,.$$

We also take the tensors χ as depending on s only. The Euler-Lagrange equations for the Lagrangian density $\sqrt{L_\nu}$ are not independent because $\sqrt{L_\nu}$ is a homogeneous function of degree 1 and thus satisfies an additional homogeneous differential equation. Then, it is well-known that the variational problem for S_1 is equivalent to consider the variation of the action S_2 defined by

$$S_2 = \int_0^\ell \nu(U(s), U(s))\, ds \equiv \int_0^\ell L_\nu\, ds\,,$$

but constrained by the condition $L_\nu = 1$. In this case, this shows that L_ν must be considered, firstly, with an associated Lagrange multiplier, namely a mass, and secondly, its explicit expression with respect to U will appear only in the variational calculus. In the weak fields limit, we obtain:

$$L_\nu = \|U\|^2 + 2 \sum_{j,k=1}^{n} \omega_{kj}\, \chi^k\, U^j\, .\, \sum_{i=1}^{n} \mathcal{A}_i\, U^i + 2 \sum_{j,k,h=1}^{n} \chi^{k,h}\, \omega_{kj}\, U^j\, .\, \sum_{i=1}^{n} U^i\, \mathcal{B}_{i,h}\,. \tag{51}$$

From the latter relation, we can deduce a few physical consequences among others. On the one hand, if we assume that

$$\mathcal{C}^h(\chi, U) \equiv \sum_{j,k=1}^{n} \omega_{kj} \chi^{k,h} U^j \equiv \zeta^h\,, \tag{52a}$$

$$\mathcal{C}^0(\chi, U) \equiv \sum_{k,j=1}^{n} \omega_{kj} \chi^{k} U^j \equiv \zeta_0\,, \tag{52b}$$

then we recover in (51), up to some suitable constants, the Lagrangian density for a particle, with the *velocity n-vector* U ($\|U\|^2 = 1$), embeded in an external electromagnetic field. But also from the relation (52b) we find *"a generalized Thomas precession"* if the tensor (χ^k) is ascribed (up to a suitable constant for units) to a *"polarization n-vector"* [47, p. 270] *"dressing"* the particle (e.g. the spin of an electron). This generalized precession could give a possible origin for the creation of anyons in high-T_c superconductors [48] and might be an alternative to Chern-Simon theory. Also, the tensor $(\chi^{k,h})$ might be a *polarization tensor* of some matter and the particle would be "dressed" with this kind of polarization.

More generally, the Euler-Lagrange equations associated to S_2 would define a system of geodesic equations with Riemann-Christoffel symbols Γ associated to ν and such that (with $\nu^{ij} \simeq \omega^{ij}$ at first order and assuming the χ's being constants)

$$\frac{dU^r}{ds} = -\sum_{j,k=1}^{n} \Gamma^r_{jk}\, U^j\, U^k + \xi_0 \sum_{i,k=1}^{n} \omega^{kr}\, \mathcal{F}_{ki}\, U^i\,, \tag{53}$$

with

$$\Gamma^r_{jk} = \frac{1}{2}\left(\chi^r(\partial_k\mathcal{A}_j + \partial_j\mathcal{A}_k) + \sum_{h=1}^{n}\chi^{r,h}\left(\partial_j\mathcal{B}_{k,h} + \partial_k\mathcal{B}_{j,h}\right)\right) ,$$

and $\mathcal{A}$ and $\mathcal{B}$ satisfying the first and second sets of differential equations (compare (53) with the analogous equation (6.9″) in [6] but with different Riemann-Christoffel symbols). Then the tensor Γ would be associated to gravitational fields, also providing other physical interpretations for the tensors χ.

Nevertheless, equations (53) are deduced irrespective of the conditions (52). Taking them into account would lead to a modification of the action S_2 resulting of the introduction of Lagrange multipliers λ_0 and λ_k ($k = 1, \cdots, n$) in the Lagrangian density definition. We would then define a new action:

$$S_2 = \int_0^{\ell}\left\{m\|U\|^2 + \sum_{i=1}^{n}\epsilon_i\, U^i - \sum_{k=0}^{n}\lambda_k\,\mathcal{C}^k(\chi, U)\right\} ds\,.$$

The associated Euler-Lagrange equations would be analogous to (53), but with additional terms coming from the precession. Moreover, since we have the constraint $\|U\|^2 = 1$, we need a new Lagrangian multiplier denoted by m. That also means we do computations on the projective spaces $H(1, n-1)$. From this point of view, the Lagrange multipliers appear to be non-homogeneous coordinates of these projective spaces.

Then the variational calculus would also lead to additional precession equations giving torsion as mentioned in comments of chapter 1. Again, torsion is not related to unification but to parallel transports on manifolds which is a well-known geometrical fact [49]. Hence, the existence of such precession phenomenon for a spin n-vector (χ^k) would be correlated with the existence of linear ODE for a charged particle of charge ξ_0 interacting with an electromagnetic field. Otherwise without (52b) the ODE's would be non-linear and there wouldn't be any kind of precession of any spin n-vector.

Consequently the motion defined by the second term in the r.h.s. of (53) for a spinning charged particle would just be, in this model, a point of view resulting from an implicit separation of rotations and translations degrees of freedom achieved by the specialized (sensitive to particular subgroups of the symmetry group of motions) experimental apparatus in $T_{p_0}\mathcal{M}$. The latter separation would insure either some simplicity i.e. linearity, or, since the measurements are achieved in $T_{p_0}\mathcal{M}$, that the equations of motion are associated (*via* some kind of projections inherent to implicit dynamical constraints due to the experimental measurement process, fixing, for instance, ξ_0 to a constant) to linear representations of tangent actions of the Lorentz Lie group on $T_{p_0}\mathcal{M}$. In the latter case, one could say, a somewhat provocative way, that special relativity covariance would have to be satisfied, *as much as possible*, by physical laws. This "reduction" to linearity can't be done on the first summation in the r.h.s. of (53), which must be left quadratic contrarily to the second one, since the Riemann-Christoffel symbols can't be defined equivariantly (other arguments can be found in [6]).

To conclude, equations (53) would provide us with another interpretation of spin (the χ's) as an object allowing moving particles to generate effective spacetime deformations, as "wakes" for instance.

5. Conclusion: images set back from their objects

In fact in this work, using the Pfaff systems theory and the Spencer theory of differential equations, we studied the formal solutions of the conformal Lie system with respect to the Poincaré one. More precisely, we determined the difference between these two sets of formal solutions. We gave a description of a "relative" set of PDE, namely the "c system" which defines a deployment from the Poincaré Lie pseudogroup to a sub-pseudogroup of the conformal Lie pseudogroup. We studied these two systems of Lie equations because of their occurrence in physics, particularly in electromagnetism as well as in Einsteinian relativity.

On the basis of this concept of deployment, we made the assumption that the unfolding is related to the existence of two kind of spacetimes, namely: the substratum or striated spacetime $\mathcal{S}$ from which the 4D-ocean or smooth spacetime $\mathcal{M}$ is unfolded. We recall that not all the given metrics on $\mathcal{S}$ can be admissible in order to have such spacetime $\mathcal{M}$. In the case of a substratum spacetime $\mathcal{S}$ endowed with an appropriate S-admissible metric allowing for unfolding, we assumed that $\mathcal{S}$ is equivariant with respect to the conformal and Poincaré pseudogroups and set its Riemannian scalar curvature to a constant $n(n-1)k_0$ and its Weyl tensor to zero. Then the deployment evolution can be trivial or not depending on occurences of spacetime singularities (of the potentials of interaction forces) parametrizing or dating what can be considered somehow as a kind of spacetime history. The potentials of interactions are built out of a particular relative Spencer differential sequence associated to the "c system" and describing smooth deformations of $\mathcal{S}$. Then a "local" metric or "ship-metric" defined on a moving tangent spacetime ship $T_{p_0}\mathcal{M}$ of the unfolded spacetime $\mathcal{M}$ is constructed out of the S-admissible substratum metric and of the deformation potentials. The spacetime ships dynamics are given by a system of PDE satisfied by their Lorentzian velocity n-vectors U, exhibiting both classical electrodynamic and "local" (since ship-metrics are local) geodesic navigation in spacetime endowed with gravitation.

This is a "classical approach" and quantization doesn't seem to appear. Nevertheless, in the present framework, a few physical exotic quantum effects and experiments might be revisited from a classical viewpoint: for instance the Aharanov-Böhm effect and the "one plate dynamical Casimir effect". Indeed, the first summation in the r.h.s. of (53) could give a classical effect similar to the Aharanov-Böhm quantum one [50] and could be another origin for the creation of anyons in superconductors. Usually the symmetric part of the 4-gradient of $\mathcal{A}$ appearing in the expression for Γ is never taken into account in classical physics. But, and it is a deep problem, it is contradictory with quantum mechanics since in that case the 4-vector $\mathcal{A}$ (and not its 4-gradient) is involved in the Shrödinger equation for electromagnetic interactions. In other words, $\mathcal{A}$ defines a set of physical observables and, as a consequence, the symmetric part of its 4-gradient would have to define also other physical observables. These latter never appear in classical electrodynamics and in usual classical physics. In some way, the occurence of this symmetric part in Γ is much more coherent with quantum mechanics and would not be meaningless.

Also, going on again in a non-rigorous metaphoric description, could the one plate (linear) dynamical Casimir effect [51, 52] be analogous to the *cavitation* of a paddle or of a fan blade of a screw-propeller in a spacetime 4D-ocean, with steam bubbles being made

of photons and/or gravitons ? This cavitation effect would occur as early as the motion of the uncharged metallic plate wouldn't be an uniformly accelerated one. Then the conformal spacetime symmetry would be broken and occurence of bubbles of "electromagnetic and gravitational steam" dressing the plate would restore only the conformal symmetry when considering the full resulting system. In the same vein, gravitational waves would be radiated only by non-uniformly accelerated massive bodies. Also, on the basis of the previous Casimir effect with its screw-propeller metaphor, the case of a non-uniformly accelerated uncharged conductive rotating disk would have to be experimentally considered in order to investigate new forms and conditions of electromagnetic and/or gravitational radiations. This would be a one plate *rotational* dynamical Casimir effect.

To finish on a more philosophical note, which is rather unavoidable since we are concerned with spacetime structure, the model we present is strongly related to the H. Bergson one in his well-known book about the A. Einstein relativity: "Durée et Simultanéité" [53]. It seems that H. Bergson was misunderstood and opposed to Einstein relativity whereas exactly the converse appears to be true. In fact H. Bergson claimed 1) that A. Einstein relativity is a confirmation of the existence of a universal time (i.e. history and not duration) because of an implicit reciprocity involved in the relativity principles and 2) that A. Einstein gave the keys to interpret some "appearances" resulting from observations between *Galilean frames* (which involves to consider the Poincaré pseudogroup only). As a particular result the twin paradox of M. Langevin doesn't exist any more as H. Bergson shown in appendix III of [53]. In the framework of the model we present, this can be easily shown by making integrations along universe lines by using the results at the end of the chapter 3 about the transformation laws between Galilean frames with velocity n-vectors $\dot{x}$ and $\dot{\hat{x}}^i = \sum_{j=1}^n \hat{f}^i_j \, \dot{x}^j$. The "time" integrals, i.e. the actions S_1, are coordinate (topological) invariant and equal. In other words, following a kind of G. Berkeley philosophy of eyesight strongly related to our Quattrocento painters metaphor evoked in the first chapter, the "physical reality" would have to be interpreted with the help of the Lorentz transformations. Indeed special relativity measurements would be achieved somehow only on "observed images set back from their objects and following them like their shadows", and as far away from their objects and slowly varying as the relative velocity between their objects and the frames of measurements increase. This is a Doppler-Fizeau effect.

We can also point out some recent reflexions of G. 't Hooft about obstacles on the way towards the quantization of space, time and matter [54]. His theory of "ontological states" involved in his approach of "deterministic quantization", strongly requires such "universal time" and a description of spacetime as a fluid (his own words) as well as its relationship with the principle of coordinate invariance. A very analogous approach of the 't Hooft one for the time irreversibility, can be considered with the Prigogine concept of thermodynamical time operator, in the framework of the Kolmogorov flows for PDE [55]. It would be applied to the non-linear "c system". Thus, there would be two kind of times: a unique global cosmic time (as in chapter 1) orienting local thermodynamical ones, the same way as the terrestrial magnetic field orients the magnetic needles of compasses.

References

[1] W. G. Unruh, Phys. Rev. D **51**, 2827 (1995).

[2] A. kostro, in: *Proceedings of the Physical Interpretations of Relativity Theory (P.I.R.T.) Conference* (p. 206, London, 1994).
[3] J. Lévy, *Relativité et Substratum Cosmique* (Librairie LAVOISIER, TECH & DOC, Cachan, France, 1996).
[4] H. Bateman, *Proc. Lond. Math. Soc.* **8** (1910), 223.
[5] E. Cunningham, *Proc. Lond. Math. Soc.* **8** (1909), 77.
[6] T. Fulton, F. Rohrlich, L. Witten, *Rev. of Mod. Phys.* **34** (1962), 442.
[7] G. Deleuze, F. Guattari, "Mille Plateaux," Les Éditions de Minuits, Paris, 1980.
[8] M. Ghins and T. Budden, *Stud. Hist. Phil. Mod. Phys.* **32** (2001), 33.
[9] H. Weyl, *in* "Selecta Hermann Weyl," p. 249, Birkhäuser Verlag, Basel und Stuttgart, 1956.
[10] D. Ivanenko and G. Saradanashvily, *Physics Reports* **94** (1983), 1.
[11] J.-M. Souriau, *Ann. Inst. Henri Poincaré A* Vol. XX (1974), 315.
[12] H. Weyl, *in* "The Dawning of Gauge Theory," (O'Raifeartaigh, Ed.), Princeton University Press, Princeton (N.J.), 1997.
[13] ———, "Space-Time Matter," Dover publishing, 1952.
[14] M. O. Katanaev and I. V. Volovich, *Ann. Phys.* **216** (1992), 1.
[15] H. Kleinert, "Gauge Fields in Condensed Matter," Vol. II, World Scientific, 1989.
[16] J.-F. Pommaret, "Partial Differential Equations and Group Theory, New Perspectives for Applications," Mathematics and Its Applications, Vol. 293, Kluwer Academic Publishers, Dordrecht-Boston-London, 1994.
[17] S. Deser, R. Jackiw, G. 't Hooft, *Ann. Phys.* **152** (1984), 220.
[18] J. R. Gott, M. Alpert, *Gen. Rel. Grav.* **16** (1984), 243.
[19] S. Giddings, J. Abbott, K. Kuchař *Gen. Rel. Grav.* **16** (1984), 751.
[20] D. C. Spencer, *Bull. Amer. Math. Soc.* **75** (1969), 179.
[21] J. Gasqui and H. Goldschmidt, "Déformations Infinitésimales des Structures Conformes Plates," Progress in Mathematics, Vol. 52, Birkhaeuser, Boston-Basel-Stuttgart, 1984.
[22] K. Yano, "Integral Formulas in Riemannian Geometry," Marcel Dekker Inc., 1970.
[23] J. Gasqui, *Ann. Scient. Éc. Norm. Sup.* 4^{e} série (1979), 101.
[24] B. Malgrange, *J. Diff. Geom.* **6** (1972), 503.
[25] ———, *J. Diff. Geom.* **7** (1972), 117.
[26] A. Kumpera and D. C. Spencer, "Lie Equations, Vol. I: General Theory," Annals of Mathematics Studies, Vol. 73, Princeton University Press, Princeton (N.J.), 1972.
[27] M. A. Akivis, V. V. Goldberg, "Conformal Differential Geometry and its Generalizations," Pure and Applied Mathematical Series, John Wiley & Sons, Inc., New York, 1996.
[28] L. Page, *Phys. Rev.* **49** (1936), 254; L. Page, *Phys. Rev.* **49** (1936), 946; L. Page, N. I. Adams, *Phys. Rev.* **49** (1936), 466.
[29] J. Haantjes, *Proc. Akad. Wet. Amsterdam* **43** (1940), 1288.
[30] M. Abraham, *Jb. Radioakt.* **11** (1914), 470.
[31] C. W. Misner, K. S. Thorne, and J. A. Wheeler, "Gravitation," W. H. Freeman and Company, San Francisco, 1973.
[32] W. M. Tulczyjew, *in* "Geometrodynamics Proceedings," (1983), p. 91, Tecnoprint, Bologna, 1984.
[33] L. De Broglie, "La Thermodynamique de la Particule Isolée," Gauthier-Villars Ed., Paris, 1964.
[34] A. Einstein, *Jb. Radioakt. Electronik* **4** (1907), 411.
[35] M. von Laue, *Ann. der Phys.* **23** (1907), 989.
[36] M. Planck, *Ann. der Phys.* **76** (1908), 1.
[37] H. Z. Ott, *Phys.* **175** (1963), 70.
[38] H. Arzeliès, *Il Nuovo Cim.* **35** (1965), 792.
[39] C. Møller, "The Theory of Relativity," Oxford University Press, 2d. ed., 1971.
[40] E. Cartan, "Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques," Hermann, Paris, 1945.
[41] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, "Exterior Differential Systems," MSRI Publications, Vol. 18, Springer-Verlag, New York, 1991.

[42] C. J. Rust, G. J. Reid, A. D. Wittkopf, *in* "Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation," p. 105, ACM Press, New York, 1999.
[43] J. Haantjes, *Proc. Akad. Wet. Amsterdam* **44** (1941), 814.
[44] T. Tsujishita, "Formal Geometry of Systems of Differential Equations," Sugaku Expositions, A.M.S., 1990.
[45] I. Anderson, G. Thompson, "The Inverse Problem of the Calculus of Variations for Ordinary Differential Equations," Memoirs of the A.M.S., no. 473, 1992.
[46] I. G. Lisle, G. J. Reid, *in* "Geometric Approaches to Differential Equations" (P. J. Vassiliou, I. G. Lisle, Eds.), p. 116, Cambridge University Press, 1999.
[47] H. Bacry, "Leçons sur la Théorie des Groupes et les Symétries des Particules Élémentaires," Gordon and Breach, Paris, 1967.
[48] J. L. Rubin, *Ferroelectrics* **161** (1994), 335.
[49] J. Dieudonné, "Éléments d'analyse," Tome IV, Gauthier-Villars Éditeur, Paris, 1971.
[50] Y. Aharanov, D. Böhm, *Phys. Rev.* **115** (1959), 125.
[51] G. T. Moore, *J. Math. Phys.* **11** (1970), 2679.
[52] S. A. Fulling, P. C. W. Davies, *Proc. R. Soc. London* **A 348** (1976), 393.
[53] H. Bergson, "Durée et Simultanéité," Quadrige/Presse Universitaires de France, Paris, 1998.
[54] G. 't Hooft, *Stud. Hist. Phil. Mod. Phys.* **32** (2001), 157.
[55] B. Misra, *J. Stat. Phys.* **5/6** (1987), 1295.